The University of Texas at Austin
Dept. of Electrical and Computer Engineering
Midterm #1 ***Solution Set 6.0***

Date: October 4, 2018 Course: EE 313 Evans

Name: **Take** **Five**
Last, First

- The exam is scheduled to last 75 minutes.
- Open books and open notes. You may refer to your homework assignments and the homework solution sets.
- Calculators are allowed.
- You may use any standalone computer system, i.e. one that is not connected to a network.
- ***Please disable all wireless connections on your calculator(s) and computer system(s).***
- Please turn off all cell phones.
- No headphones are allowed.
- All work should be performed on the midterm exam. If more space is needed, then use the backs of the pages.
- **Fully justify your answers.** If you decide to quote text from a source, please give the quote, page number and source citation**.**

	Problem	*Point Value*	*Your score*	*Topic*
Joe Morello	1	18		Sampling Sinusoids
Eugene Wright	2	18		Harmonics
Paul Desmond	3	24		Fourier Series Analysis
Dave Brubeck	4	24		Undersampling
Teo Macero	5	16		Potpourri
	Total	100		

Problem 1.1 *Sampling Sinusoids*. 18 points.

SPFirst *Sec. 4-1 & 4-2*	*Lecture slides 5-5 & 5-6*
Homework Prob. 3.2	*Mini-Project #1 Prob. 3.2*
	Lecture slides 6-4 to 6-7

Consider the sinusoidal signal $x(t) = \cos(2\pi f_0 t + \theta)$ for continuous-time frequency f_0 in Hz.

We then sample $x(t)$ at a sampling rate f_s in Hz to produce a discrete-time signal $x[n]$.

(a) Derive the formula for $x[n]$ by sampling $x(t)$. *6 points.*

$$x[n] = x(t)|_{t=nT_s} = \cos(2\pi f_0 (nT_s) + \theta) = \cos(2\pi f_0 T_s n + \theta) = \cos\left(2\pi \frac{f_0}{f_s} n + \theta\right)$$

$$x[n] = \cos(\hat{\omega}_0 n + \theta)$$

(b) Based on your answer in part (a), give a formula for the discrete-time frequency $\hat{\omega}_0$ in of $x[n]$ in terms of the continuous-time frequency f_0 and sampling rate f_s. Units of $\hat{\omega}_0$ are in rad/sample. *6 points.*

$$\hat{\omega}_0 = 2\pi \frac{f_0}{f_s}$$

(c) For continuous-time frequency f_0 = 440 Hz and sampling rate f_s = 8000 Hz,

i. What is the smallest discrete-time period for $x[n]$? Why? *3 points.*

$$\hat{\omega}_0 = 2\pi \frac{f_0}{f_s} = 2\pi \frac{440 \text{ Hz}}{8000 \text{ Hz}} = 2\pi \frac{11}{200} = 2\pi \frac{N}{L}$$

where N and L are relatively prime integers. From Handout D *Discrete-Time Periodicity*, a discrete-time signal $x[n]$ has period N_0 if $x[n + N_0] = x[n]$ for all n.

$$x[n] = \cos(\hat{\omega}_0 n + \theta) = \cos\left(2\pi \frac{N}{L} n + \theta\right)$$

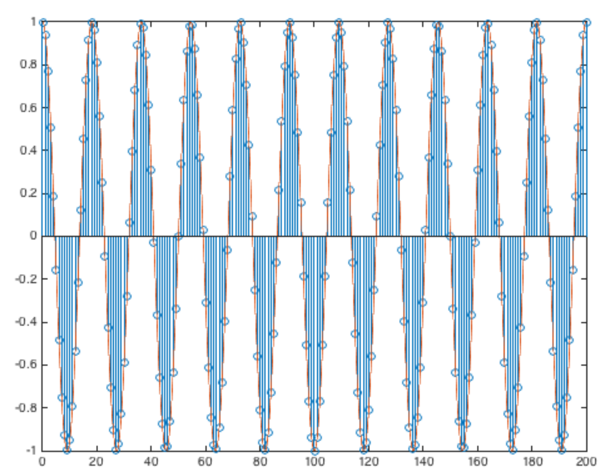
$$x[n + N_0] = \cos\left(2\pi \frac{N}{L}(n + N_0) + \theta\right) = \cos\left(2\pi \frac{N}{L} n + 2\pi \frac{N}{L} N_0 + \theta\right) = x[n]$$

if N_0 is an integer multiple of L. The smallest period occurs when $N_0 = L$.

ii. How many continuous-time periods of $x(t)$ are in the smallest discrete-time period of $x[n]$? Why? *3 points.*

From Handout D *Discrete-Time Periodicity*, there are N continuous-time periods of $x(t)$ in the smallest discrete-time period of $x[n]$.

```
% MATLAB Code (not asked)
% Discrete-time period of 200 samples
% contains 11 continuous-time periods
fs = 8000;
Ts = 1/fs;
f0 = 440;
wHat = 2*pi*f0/fs;
N0 = 200;
n = 0 : N0;   % 201 Samples
yofn = cos(wHat*n);
t = 0 : 0.01 : N0;
yoft = cos(wHat*t);
figure;
stem(n, yofn);
hold;
plot(t, yoft)
```



Problem 1.2 *Harmonics.* 18 points.

(a) ***Virtual Bass.*** Human auditory systems have the ability to perceive a frequency when the frequency is not present but many of its harmonics are present. Consider a less capable audio speaker, which can only play frequencies between 200 Hz and 1000 Hz. The speaker plays an audio clip and produces principal frequencies of 210 Hz, 315 Hz, 420 Hz, 525 Hz, 630 Hz, 735 Hz, 840 Hz, and 945 Hz *at the same time*. What additional principal frequency could a human listener perceive? *6 points*.

SPFirst Sec. 3-3
Lecture slide 3-4
Homework Prob. 2.1

Greatest common divisor among the principal frequencies is 105 Hz; principal frequencies 210, 315, ... 945 Hz are harmonics of 105 Hz.

A human might perceive 105 Hz even though it isn't present.

Note: **This property can be exploited to generate sub-wolfer bass frequencies (20-200 Hz) without having the ability to play these frequencies by generating wolfer frequencies (200-2000 Hz) with the right harmonic structure [2]. This property was also used in the original wired telephone service that transmitted frequencies in the 300-3300 Hz range. Human voice pitch frequencies are below 300 Hz, but the human auditory system would perceive the pitch frequency due to the harmonic structure in voice utterances, such vowel sounds [1][2]. The telephone companies reserved 0-200 Hz for their own signaling purposes. Also, using audible frequencies in the 300-3300 Hz range would have simplified the speaker design.**

(b) ***Nonlinearity Effect.*** Consider the signal $x(t) = \cos^3(2\pi f_0 t)$. Write the signal using the Fourier series synthesis formula

SPFirst Sec. 3-3 & 3-6 | *Lecture slides 3-4 to 3-14*
Homework Prob. 2.4

$$x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi(kf_0)t}$$

i. What is the value of N? *3 points.*

We can expand the cosine signal using the inverse Euler relationship as follows:

$$\boldsymbol{x(t) = \cos^3(2\pi f_0 t) = \left(\frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2}\right)^3 = \frac{1}{8}\left(e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}\right)^3}$$

and then expand the term $(a+b)^3$ using a binomial expansion:

$$\boldsymbol{(a+b)^3 = (a+b)(a+b)^2 = (a+b)(a^2+2ab+b^2) = a^3 + 3ba^2 + 3ab^2 + b^3}$$

The binomial expansion has coefficients 1, 2, 1 for second-order and 1, 3, 3, 1 for third order. Using the binomial expansion for third order, $\underline{N = 3}$:

$$\boldsymbol{x(t) = \frac{1}{8}e^{j2\pi(3f_0)t} + \frac{3}{8}e^{j2\pi f_0 t} + \frac{3}{8}e^{-j2\pi f_0 t} + \frac{1}{8}e^{-j2\pi(3f_0)t}}$$

ii. Give the values of all of the Fourier series coefficients a_k for $k = -N, \ldots, 0, \ldots, N$. *9 points.*

$$\boldsymbol{a_3 = a_{-3} = \frac{1}{8} \text{ and } a_2 = a_{-2} = 0 \text{ and } a_1 = a_{-1} = \frac{3}{8} \text{ and } a_0 = 0}$$

References for Problem 1.2(a)

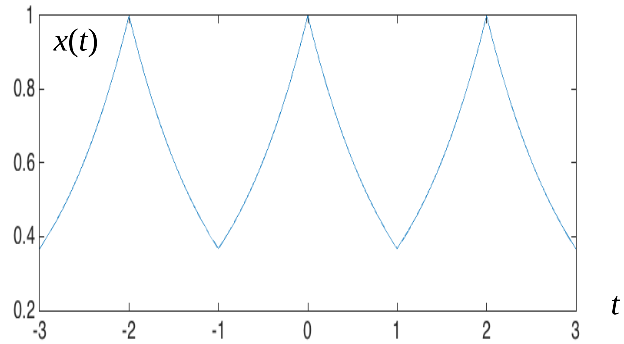
[1] Principal frequencies in this problem roughly correspond to the English phoneme 'aw'
[2] "Missing fundamental", Wikipedia, accessed Oct. 11, 2018.

Problem 1.3 *Fourier Series Analysis.* 24 points.

A periodic signal $x(t)$ is defined over its fundamental period of duration T_0 as

$$e^{t} \quad \text{for} -\frac{T_0}{2} \leq t < 0$$
$$e^{-t} \quad \text{for } 0 \leq t < \frac{T_0}{2}$$

and plotted over three fundamental periods on the right.



Compute the Fourier series coefficients using

$$a_k = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t) e^{-j2\pi(kf_0)t}\, dt$$

by answering the questions below.

SPFirst Sec. 2-3

Homework Prob. 1.1

Lecture slides 1-11 & 1-12

(a) What is the value of T_0? How did you compute it? *6 points.*

Solution #1: **Compute the distance in time from a peak to the next peak. T_0 = 2 seconds.**

Solution #2: **Compute the distance in time from a valley to the next valley. T_0 = 2 seconds.**

Solution #3: **Over the 6 seconds plotted, three fundamental periods occur. T_0 = 2 seconds.**

(b) What is the value of a_0? What does the value of a_0 represent in $x(t)$? *6 points.*

$$a_0 = \frac{1}{2}\int_{-1}^{1} x(t)\,dt = \frac{1}{2}\int_{-1}^{0} e^{t}\,dt + \frac{1}{2}\int_{0}^{1} e^{-t}\,dt$$

$$a_0 = \frac{1}{2}\left[e^{t}\right]_{-1}^{0} + \frac{1}{2}\left[-e^{-t}\right]_{0}^{1} = \frac{1}{2}\left(1 - e^{-1}\right) - \frac{1}{2}\left(e^{-1} - 1\right) = 1 - e^{-1}$$

a_0 represents the average value of $x(t)$ over the fundamental period.

SPFirst Sec. 3-4

Homework Prob. 3.1(d)

Lecture slides 3-8 to 3-11

(c) Give a formula for a_k. *12 points.*

Solution #1: **With T_0 = 2 s, $f_0 = 1/T_0$ = 0.5 Hz:**

$$a_k = \frac{1}{T_0}\int_{-\frac{1}{2}T_0}^{\frac{1}{2}T_0} x(t) e^{-j2\pi k f_0 t}\,dt = \frac{1}{2}\int_{-1}^{1} x(t) e^{-j\pi k t}\,dt = \frac{1}{2}\int_{-1}^{0} e^{t} e^{-j\pi k t}\,dt + \frac{1}{2}\int_{0}^{1} e^{-t} e^{-j\pi k t}\,dt$$

$$a_k = \frac{1}{2}\int_{-1}^{0} e^{(1-j\pi k)t}\,dt + \frac{1}{2}\int_{0}^{1} e^{(-1-j\pi k)t}\,dt = \frac{1}{2}\frac{1}{(1-j\pi k)}\left[e^{(1-j\pi k)t}\right]_{-1}^{0} - \frac{1}{2}\frac{1}{(1+j\pi k)}\left[e^{-(1+j\pi k)t}\right]_{0}^{1}$$

$$a_k = \frac{1}{2}\frac{1}{(1-j\pi k)}\left(1 - e^{-1+j\pi k}\right) - \frac{1}{2}\frac{1}{(1+j\pi k)}\left(e^{-1-j\pi k} - 1\right)$$

Solution #2: **Use the even symmetry of $x(t)$ about $t = 0$ in one fundamental period to compute the integral from 0 to 1 and then double the result.**

***Note:* $x(t)$ could represent an amplitude in time when playing a musical note with "attack" from -1 to 0 s and "release" from 0 to 1 s. A related example is charging/discharging of an RC circuit. $x(t)$ is the voltage across the capacitor: $1 - \exp(-(t+1)/\tau)$ for $-1 < t \leq 0$ and $\exp(-t/\tau)$ for $0 < t \leq 1$, where τ is the time constant (e.g. 0.01). See https://www.electronics-tutorials.ws/rc/rc_1.html.**

Problem 1.4. *Undersampling.* 24 points**.**

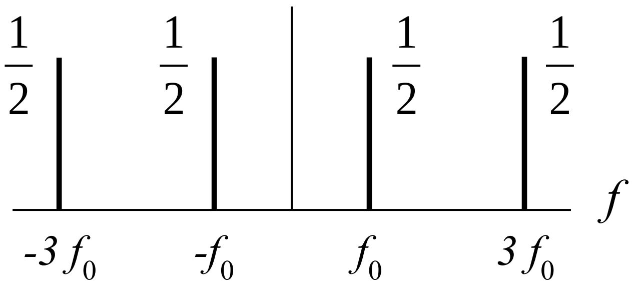
In certain systems, the main alternating power frequency f_0 in Hz and its odd harmonics ($3f_0$, $5f_0$, etc.) cause interference. This problem will explore the effect of undersampling on the interference.

For simplicity, model the interference signal $g(t)$ as having the first and third harmonics of f_0:

$$g(t) = \cos(2\,\pi\,f_0\,t) + \cos(2\,\pi\,(3\,f_0)\,t)$$

Let $f_0 = 60$ Hz as it is in the US. (In many other countries, $f_0 = 50$ Hz.)

(a) Draw the spectrum of $g(t)$. Include positive and negative continuous-time frequencies. *6 points.*



SPFirst Chap. 3 Intro & Sec. 3-1

Lecture slides 2-10 to 2-12; 3-2 to 3-4

Homework Prob. 2.2

(b) When sampling $g(t)$ at a sampling rate $f_s = 2f_0 = 120$ Hz obtain a formula for $g[n]$. *3 points.*

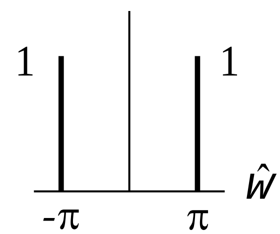
$$\boldsymbol{g[n] = g(nT_s) = g\left(\frac{n}{f_s}\right) = \cos\left(2\pi\frac{f_0}{2f_0}n\right) + \cos\left(2\pi\frac{3f_0}{2f_0}n\right) = \cos(\pi n) + \cos(3\pi n)}$$

$$\boldsymbol{g[n] = \cos(\pi n) + \cos(3\pi n - 2\pi n) = \cos(\pi n) + \cos(\pi n) = 2\cos(\pi n)}$$

Homework Prob. 3.2

Aliasing not due to folding: 3π → π and -3π → -π

(c) Draw the spectrum of $g[n]$ in part (b) for discrete-time frequencies in the interval [-π, π]. *6 points.*



SPFirst Sec. 4-1 to 4-3

Lecture slides 5-9 to 5-13; 6-4 to 6-10

(d) When sampling $g(t)$ at a sampling rate $f_s = 4f_0 = 240$ Hz obtain a formula for $g[n]$. *3 points.*

$$\boldsymbol{g[n] = g(nT_s) = g\left(\frac{n}{f_s}\right) = \cos\left(2\pi\frac{f_0}{4f_0}n\right) + \cos\left(2\pi\frac{3f_0}{4f_0}n\right) = \cos\left(\frac{\pi}{2}n\right) + \cos\left(\frac{3}{2}\pi n\right)}$$

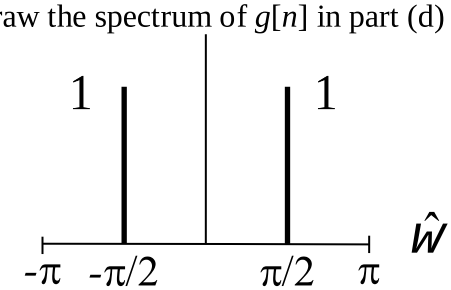
$$\boldsymbol{g[n] = \cos\left(\frac{\pi}{2}n\right) + \cos\left(\frac{3}{2}\pi n\right) = \cos\left(\frac{\pi}{2}n\right) + \cos\left(\frac{3}{2}\pi n - 2\pi n\right)}$$

$$\boldsymbol{g[n] = \cos\left(\frac{\pi}{2}n\right) + \cos\left(-\frac{\pi}{2}n\right) = \cos\left(\frac{\pi}{2}n\right) + \cos\left(\frac{\pi}{2}n\right) = 2\ \cos\left(\frac{\pi}{2}n\right)}$$

Aliasing due to folding: (3/2)π → -(1/2)π and -(3/2)π → (1/2)π

Homework Prob. 3.2

(e) Draw the spectrum of $g[n]$ in part (d) for discrete-time frequencies in the interval [-π, π]. *6 points.*



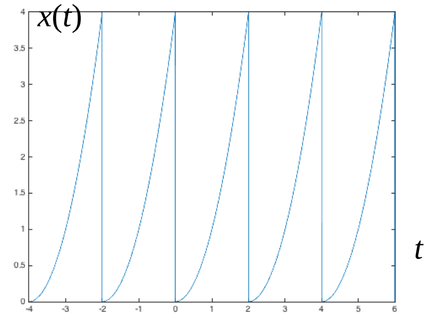
SPFirst Sec. 4-1 to 4-3

Lecture slides 5-9 to 5-13; 6-4 to 6-10

Problem 1.5. *Potpourri*. 16 points.

(a) Consider the periodic signal shown on the right with a fundamental period of $T_0 = 2$ seconds. Over one fundamental period,

$$x(t) = t^2$$



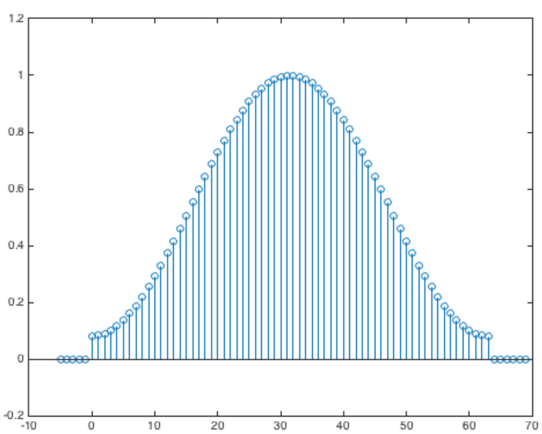
If we keep a large but finite number of Fourier series coefficients, explain whether or not the Fourier synthesis will suffer from Gibbs phenomenon.

In your answer, please explain what Gibbs phenomenon is. *8 points*.

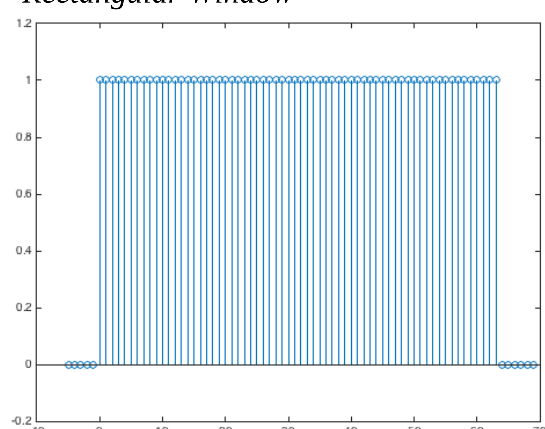
Yes, $x(t)$ will suffer from Gibbs' phenomenon because it has amplitude discontinuities that occur at even time instances (… -4s, -2s, 0s, 2s, 4s, …). Gibbs' phenomenon occurs at/near each amplitude discontinuity when using a finite number of Fourier series synthesis terms. An amplitude discontinuity could occur at or inside the boundaries of the fundamental period, depending what interval of time is chosen for the fundamental period. No matter how many finite terms are used for Fourier series synthesis, oscillation will be seen at both amplitude values across the amplitude discontinuity. This oscillation will lead to a worst-case error between the Fourier series synthesis and the original signal on either side of the amplitude discontinuity of no less than ±9% regardless of how many finite terms are used (*SPFirst* Sec. 3-6.6). Fig. 3-17 on page 54 of *SPFirst* shows Gibbs' phenomenon for the square wave. Gibbs' phenomenon does not occur for a periodic signal that does not have an amplitude discontinuity, even if the periodic signal has a discontinuity in derivative at one of more points. For example, a triangular wave does not suffer from Gibbs' phenomenon (*SPFirst* Sec. 3-6.6). Gibbs' phenomenon does not exist if an infinite number of terms is used.

(b) In spectrogram calculations, explain why it is generally advantageous to use a Hamming window instead of a rectangular window to weight the amplitudes in each segment (block) of samples. Stem plots of Hamming and rectangular windows of length 64 samples are given below. *8 points*.

Hamming Window

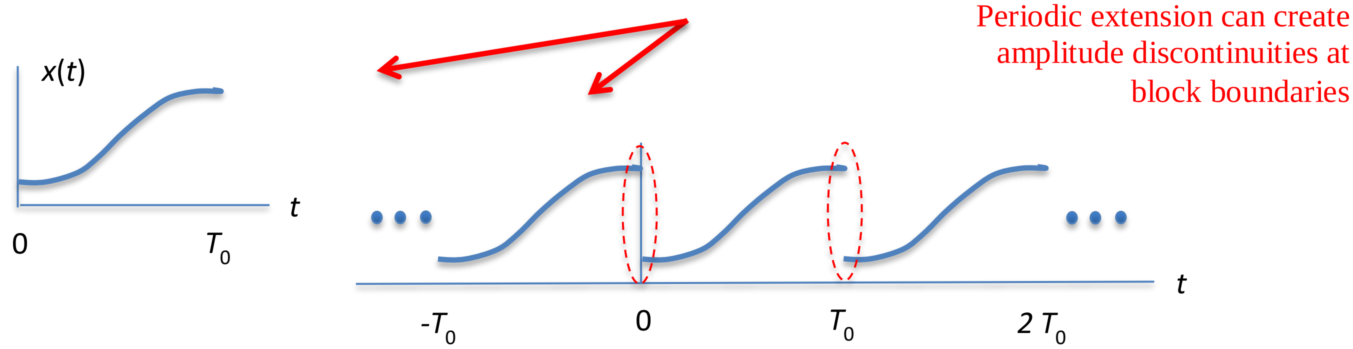


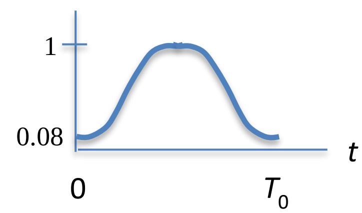
Rectangular Window



A spectrogram would multiply each block of samples by a window and then compute a discrete-time Fourier series, which assumes that the block of samples is the fundamental period of a periodic signal. Periodic replication of the block of samples can create amplitude discontinuities at block boundaries, which will create artificial high frequencies and lead to

Gibbs' phenomenon. A Hamming window tapers amplitudes of samples at/near block boundaries to reduce artifacts caused by amplitude discontinuities at the boundaries.





Hamming window will reduce the amplitudes at/near block boundaries to reduce strength of artificially created high frequencies

Definition of a Hamming window $w[n]$ with N samples where $0 \le n \le N\text{-}1$:

$$w[n] = 0.54 - 0.46 \cos\left(2\pi \frac{n}{N-1}\right)$$

MATLAB Code to Generate the Plots on Midterm #1

```
% 1.3
T0 = 2;
f0 = 2;
fs = 1000*f0;
Ts = 1/fs;
Np = 3;
t = -Np*T0/2 : Ts : Np*T0/2;
tmod = mod(t - T0/2, T0) - T0/2;
x = exp( -abs(tmod) );
plot(t, x);
% Manually changed the font size to 20pt
% using the plot manager window

% 1.5(a) Periodic signal
T0 = 2;
f0 = 1/2;
fs = 2000*f0;
Ts = 1 / fs;
t = -2*T0 : Ts : 3*T0;
x = mod(t, T0) .^ 2;
plot(t, x);

% 1.5(b) Windows
n = -5 : 69;
v = zeros(1, 75);
v(6:69) = hamming(64);
figure;
stem(n, v);
ylim( [-0.2 1.2] );

n = -5 : 69;
v = zeros(1, 75);
v(6:69) = ones(1, 64);
figure;
stem(n, v);
ylim( [-0.2 1.2] );
```